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# $\alpha$-Well-posedness for Nash equilibria and for optimization problems with Nash equilibrium constraints

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**Abstract** We present the concepts of $\alpha$-well-posedness for parametric noncooperative games and for optimization problems with constraints defined by parametric Nash equilibria. We investigate some classes of functions that ensure these types of well-posedness and the connections with $\alpha$-well-posedness for variational inequalities and optimization problems with variational inequality constraints.

**Keywords** Nash equilibrium · variational inequality · $\alpha$-well-Posedness · optimization problem with Nash equilibrium constraints

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## 1 Introduction

The importance of optimization problems whose constraints are defined by the solutions to equilibrium problems has grown up in last years.

In fact, many papers or books have been devoted to investigate such problems when the constraints are determined by optimization problems ([10, 23, 24, 28],...), variational inequalities ([13, 18, 20, 25, 33],...), or by Nash equilibria and social Nash equilibria problems [22].

Let $(X, \tau)$ be a topological space, $E_1$ and $E_2$ be two real reflexive Banach spaces.

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We consider, for $i = 1, 2$, a nonempty subset $Y_i$ of $E_i$, a function $J_i$ from $Y_1 \times Y_2$ to $R \cup \{+\infty\}$, a function $f: X \times E_1 \times E_2 \to R \cup \{+\infty\}$, a set-valued function $Q_1$ from $X$ to $Y_1$ and a set-valued function $Q_2$ from $X$ to $Y_2$.

Then, the following optimization problem with Nash equilibrium constraints (OPNEC) is considered:

$$\text{(OPNEC)} \qquad \min_{x \in X} \min_{(u_1,u_2) \in T(x)} f(x, u_1, u_2),$$

where, for every $x \in X$, $T(x)$ is the Nash equilibria set of the following parametric game in normal form:

$$\Gamma(x) = (Q_1(x), Q_2(x), J_1(x, \cdot, \cdot), J_2(x, \cdot, \cdot))$$

that is, the set of elements $(y_1, y_2) \in Q_1(x) \times Q_2(x)$ such that:

$$J_1(x, y_1, y_2) \le J_1(x, z_1, y_2) \quad \text{for all} \quad z_1 \in Q_1(x),$$
$$J_2(x, y_1, y_2) \le J_2(x, y_1, z_2) \quad \text{for all} \quad z_2 \in Q_2(x).$$

In a previous paper [22] the authors investigated, under assumptions of minimal character, the existence of solutions to such problems in reflexive Banach spaces. Here, our aim is to introduce and study a well-posedness concept for the problem (OPNEC) which is in line with the concept introduced in [9] for optimization problems with variational inequality constraints (OPVIC), overall motivated by the numerical method introduced by Fukushima [14].

We recall (see, e.g. [2]) that, if $A$ is an operator from a reflexive Banach space $E$ to its dual $E^*$ and $K$ is a nonempty, closed and convex subset of $E$, the classical variational inequality problem (VI) consists in finding a point $u_0$ such that

$$\text{(VI)} \qquad u_0 \in K \quad \text{and} \quad \langle Au_0, u_0 - v \rangle \le 0 \quad \text{for every} \quad v \in K.$$

In [21], we introduced the definitions of $\alpha$-well-posedness for VIs and for Nash equilibrium problems (NEPs) and we proved a partial result concerning the links between these two notions. We point out that the definition of $\alpha$-well-posedness for Nash equilibria had been inspired from the corresponding definition for VIs (under appropriate assumptions, a NEP is equivalent to a VI, see Proposition 2.1), which, in turns, had been inspired from the Tikhonov well-posedness for minimization problems. More precisely, it is well known that any variational inequality can be transformed into an equivalent minimization problem by using a *gap function*, that is a function $g$, such that:

(i) $g(u) \ge 0 \quad \forall\, u \in K$;
(ii) $g(u_o) = 0$ if and only if $u_o$ solves (VI).

The first gap function, introduced by Auslender [1], is the function $g$ defined on $E$ by

$$g(u) = \sup_{v \in K} \langle Au, u - v \rangle,$$

which allows to transform the VI into the equivalent optimization problem:

$$(P) \qquad \min_{u \in K} g(u).$$

In [21] a VI is defined to be well-posed whenever the corresponding minimization problem $(P)$ is well-posed in the sense of Tykhonov [11]. However, since the function $g$ is not in general differentiable, we introduced in [21] another concept of well-posedness, using the Fukushima gap function [14]:

$$g_\alpha(u) = \sup_{v \in K} \left( \langle Au, u - v \rangle - \frac{\alpha}{2} \|u - v\|^2 \right) \quad \text{for} \ \alpha > 0, \tag{1}$$

which is, in finite dimensional spaces, continuously differentiable if $A$ is continuously differentiable. More precisely, a VI is $\alpha$-well-posed if and only if the corresponding minimization problem $(P_\alpha)$ is well-posed in the sense of Tikhonov.

In the present paper, we extensively investigate the $\alpha$-well-posedness of NEPs and the links with the $\alpha$-well-posedness of VIs, both in parametric and in unparametric case.

The outline of the paper is the following: in Sect. 2, we give the notations and definitions used throughout the paper, together with some previous useful results; in Sect. 3 the $\alpha$-well-posedness of unparametric NEPs is investigated, while Sec. 4 is devoted to the parametric case, to the applications and to $\alpha$-well-posedness for the problem (OPNEC). Finally, we recall that a Social Nash equilibrium [8] (also termed Generalized Nash equilibrium [13]) of the generalized game $\Gamma = (Y_1, Y_2, J_1, J_2, Q_1, Q_2)$ is a point $(y_1, y_2) \in Y_1 \times Y_2$ such that:

$$y_1 \in Q_1(y_2) \quad \text{and} \quad J_1(y_1, y_2) \le J_1(z_1, y_2) \quad \text{for all} \quad z_1 \in Q_1(y_2),$$

$$y_2 \in Q_2(y_1) \quad \text{and} \quad J_2(y_1, y_2) \le J_2(y_1, z_2) \quad \text{for all} \quad z_2 \in Q_2(y_1),$$

where $Q_1$ and $Q_2$ are set-valued functions from $Y_2$ to $Y_1$ and from $Y_1$ to $Y_2$, respectively. The above problem, under suitable assumptions, is equivalent to a quasi-variational inequality and the study of $\alpha$-well-posedness for quasi-variational inequalities will be investigated in a further paper.

## 2 Preliminary, definitions and results

For the sake of simplicity, we limit ourselves to two-player games, but the results could be easily obtained for $n$-player games.

Assume that $E$ and $E_i$, for $i = 1, 2$, are real reflexive Banach spaces, $K$ and $Y_i$, for $i = 1, 2$, are nonempty convex and closed subsets of $E$ and $E_i$, respectively, $J_i$ is a function from $Y_1 \times Y_2$ to $R \cup \{+\infty\}$, $A$ is an operator from $E$ to the dual space $E^*$ and $\alpha$ is a non-negative real number.
First, we recall some definitions.

**Definition 2.1** [21] *A sequence $(u_n)_n$ is $\alpha$-approximating for the VI if*:

(i) $u_n \in K \ \forall \, n \in N$,
(ii) *there exists a sequence $(t_n)_n$ of real positive numbers decreasing to* 0 *such that*

$$\langle Au_n, u_n - v \rangle - \frac{\alpha}{2} \|u_n - v\|^2 \le t_n, \quad \forall v \in K, \qquad \forall \, n \in N.$$

**Definition 2.2** [21] *A* VI *is $\alpha$-well-posed* (*respectively $\alpha$-well-posed in the generalized sense*) *if it has a unique solution $u_0$ and every $\alpha$-approximating sequence $(u_n)_n$ strongly converges to $u_0$* (*respectively it has at least a solution and every $\alpha$-approximating sequence $(u_n)_n$ has a subsequence which strongly converges to a solution to VI*).

A first definition of well-posedness for VI which coincides when $\alpha = 0$ and under suitable assumptions with the concept of Definition 2.2, has been introduced by Lucchetti and Patrone (see [11]). When $E$ is finite dimensional, the set $K$ is closed and convex and the operator A is hemicontinuous and strongly monotone, the two concepts of well-posedness are equivalent to the uniqueness of the solution (see [11 p. 72, 19, Proposition 2.8]). In this paper, devoted to Nash equilibria problems, being motivated by numerical methods associated to gap functions, we will refer only to the concept introduced in Definition 2.2.

In [9] we proved that the VI is $\alpha$-well-posed if and only if the following minimization problem

$$(P_\alpha) \qquad \min_{v \in K} g_\alpha(v)$$

is Tykhonov well-posed, that is there exists a unique solution $u_o$ towards which every minimizing sequence for $g_\alpha$, defined in (1), strongly converges.

A point $\mathbf{u}_o = (u_o^1, u_o^2) \in Y_1 \times Y_2$ is a *Nash equilibrium* for the game in normal form $\Gamma = (Y_1, Y_2, J_1, J_2)$ if:

$$\text{(NEP)} \qquad \begin{array}{ll} J_1(u_1^o, u_2^o) \le J_1(y_1, u_2^o) & \forall\, y_1 \in Y_1, \\ J_2(u_1^o, u_2^o) \le J_2(u_1^o, y_2) & \forall\, y_2 \in Y_2. \end{array}$$

Concerning existence results for such problems (see, e.g. [17, 22, 32]; for stability results see [6, 29, 31]; for a duality scheme see [1]).

It is well known that, under suitable assumptions, a NEP can be transformed into a VI. In fact, the following result holds:

**Proposition 2.1** *Assume that $J_i$ is Gâteaux differentiable with respect to $u_i$, for $i = 1, 2$, on $Y_1 \times Y_2$. If $\mathbf{u}_o$ is a Nash equilibrium for $\Gamma = (Y_1, Y_2, J_1, J_2)$, then it solves the* VI *defined by the operator $A$, which associates to every $\mathbf{u} = (u_1, u_2) \in Y = Y_1 \times Y_2$ the element $A\mathbf{u}$ such that:*

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \left\langle \frac{\partial J_1}{\partial u_1}(u_1, u_2), v_1 \right\rangle + \left\langle \frac{\partial J_2}{\partial u_2}(u_1, u_2), v_2 \right\rangle.$$

If, moreover, for every $v_2 \in Y_2$ and every $v_1 \in Y_1$, $J_1(\cdot, v_2)$ and $J_2(v_1, \cdot)$ are pseudoconvex on $Y_1$ and $Y_2$, respectively, then the converse holds.

**Definition 2.3** [21] *A sequence $(\mathbf{u}_n)_n = \left(u_1^n, u_2^n\right)_n$ is $\alpha$-approximating for $(NEP)$ if there exists a sequence of real positive numbers $(t_n)_n$ decreasing to* 0 *such that:*

(1) $J_1\left(u_1^n, u_2^n\right) \le t_n + J_1\left(v_1, u_2^n\right) + \frac{\alpha}{2}\left\|u_1^n - v_1\right\|^2 \quad \forall v_1 \in Y_1\ \forall\, n \in N,$

(2) $J_2\left(u_1^n, u_2^n\right) \le t_n + J_2\left(u_1^n, v_2\right) + \frac{\alpha}{2}\left\|u_2^n - v_2\right\|^2 \quad \forall v_2 \in Y_2\ \forall\, n \in N.$

For $\alpha = 0$ one obtains the definition of asymptotically Nash equilibrium given in [26].

**Definition 2.4** [21] *The NEP is $\alpha$-well-posed* (*respectively, $\alpha$-well-posed in the generalized sense*) *if there exists a unique Nash equilibrium $\mathbf{u}_o$ and every $\alpha$-approximating sequence strongly converges to $\mathbf{u}_o$* (*respectively, if there exists at least a Nash equilibrium and every $\alpha$-approximating sequence has a subsequence which strongly converges to a Nash equilibrium*).

Finally, we recall the properties of operators and set-valued functions that will be used throughout the paper. The operator $A$ from $E$ to $E^*$ is said on follows:

- *monotone* on $K$ if $\langle Au - Av, u - v\rangle \geq 0$ for every $u \in K$ and $v \in K$;
- *pseudomonotone* on $K$ if, for every $u \in K$ and $v \in K$, $\langle Au, u - v\rangle \leq 0 \Rightarrow \langle Av, u - v\rangle \leq 0$;
- *strongly monotone* on $K$ (with modulus $\beta$) if there exists a positive number $\beta$ such that $\langle Au - Av, u - v\rangle \geq \beta \|u - v\|^2$ for every $u \in K$ and $v \in K$;
- *hemicontinuous* on $K$ if it is continuous from $E$ to $E^*$ endowed with the weak topology over every segment of $K$.

The VI has a unique solution if the operator $A$ is strongly monotone and hemicontinuous (see, e.g. [2]), while there exists at least a solution for VI if the operator $A$ is pseudomonotone and hemicontinuous and some coerciveness condition is satisfied (see, e.g. [15]). In finite dimensional spaces the existence of solutions to VIs does not need monotonicity and it is sufficient to assume that the operator is continuous and coercive (see [13], a *compendium* on numerical methods for VIs).

For the sake of completeness we recall some definitions, which will be used in the following. A set-valued function $F$ from a topological space $(X, \tau)$ to a convergence space $(Y, \sigma)$ (see [16]) is:

- (*sequentially*) $(\tau, \sigma)-$*lower semicontinuous* at $x \in X$ if, for every sequence $(x_n)_n$ $\tau-$converging to $x$ and every $y \in F(x)$, there exists a sequence $(y_n)_n$ $\sigma-$converging to $y$ such that $y_n \in F(x_n)\ \ \forall\, n \in N$;
- (*sequentially*) $(\tau, \sigma)-$*subcontinuous* at $x \in X$ if, for every sequence $(x_n)_n$ $\tau-$converging to $x$, every sequence $(y_n)_n$, such that $y_n \in F(x_n)\ \forall\, n \in N$, has a $\sigma-$convergent subsequence;
- (*sequentially*) $(\tau, \sigma)-$*closed* at $x \in X$ if for every sequence $(x_n)_n$ $\tau-$converging to $x$ and every sequence $(y_n)_n$ $\sigma-$ converging to $y$, such that $y_n \in F(x_n)\ \forall\, n \in N$, one has $y \in F(x)$.

We recall that the Hausdorff distance between two nonempty bounded subsets $H$ and $K$ of a metric space $(S, d)$, is given by:

$$h(H, K) = \max\left\{\sup_{u\in H} d(u, K),\ \sup_{w\in K} d(w, H)\right\},$$

while the noncompactness measure $\mu$ of a nonempty subset $K$ of $S$ is given by:

$$\mu(K) = \inf\left\{\varepsilon > 0 : K \subseteq \bigcup_{i=1}^{n} K_i,\ \ \mathrm{diam} K_i < \varepsilon\ \ i = 1, ..., n\right\}.$$

By the definitions of $\mu$ and $h$ one gets:

$$\mu(A) \leq \mu(K) + 2h(A, K) \tag{2}$$

for every bounded sets $A$ and $K$.

Indeed, if the above inequality does not hold for a couple of bounded sets $A$ and $K$, there exists a positive real number $\delta$ such that the set $K$ can be covered with a finite number of sets $K_i$ such that $\mathrm{diam} K_i < \delta$ and:

$$\mu(A) > \delta + 2h(A, K).$$

The sets $B(K_i, h(A, K)) = \{x \in S : d(x, K_i) < h(A, K)\}$ cover $A$ and

$$\mathrm{diam} B(K_i, h(A, K)) < \delta + 2h(A, K).$$

This implies $\mu(A) \leq \delta + 2h(A, K)$ which gives a contradiction.

## 3 $\alpha$-Well-posedness for Nash equilibrium problems

With the notations of Sect. 2, throughout this section we consider a reflexive real Banach space $E = E_1 \times E_2$, a closed and convex subset $Y = Y_1 \times Y_2$ and a nonnegative real number $\alpha$. We denote by $\mathbf{u}$ the couple $(u_1, u_2) \in Y$ and assume that the following conditions hold:

$$\text{for every } v_1 \in Y_1,\ \ J_2(v_1, \cdot) \text{ is bounded from below,}$$
$$\text{for every } v_2 \in Y_2,\ \ J_1(\cdot, v_2) \text{ is bounded from below.}$$

We recall that, if $h$ is a function from a metric space $(S, d)$ to $R \cup +\{\infty\}$, bounded from below, a classical result in well-posedness theory [11] states that the problem

$$\min_{x \in S} h(x)$$

is well-posed if and only if

$$\lim_{\varepsilon \to 0} \mathrm{diam} M_\varepsilon = 0,$$

where $M_\varepsilon$ is the set of $\varepsilon-$minimum points for $h$, that is:

$$M_\varepsilon = \left\{ x' \in K : h(x') \le \inf_{x \in K} h(x) + \varepsilon \right\}.$$

In order to study the $\alpha$-well-posedness for NEPs, we can consider two types of approximate solution sets $N_{\alpha,\varepsilon}$ and $N'_{\alpha,\varepsilon}$ (see [29], for $\alpha = 0$) defined, respectively, by:

$$N_{\alpha,\varepsilon} = \left\{ \begin{array}{r} (u_1, u_2) \in Y : J_1(u_1, u_2) \le \inf\limits_{v_1 \in Y_1} \left( J_1(v_1, u_2) + \frac{\alpha}{2} \|u_1 - v_1\|^2 \right) + \varepsilon \\ J_2(u_1, u_2) \le \inf\limits_{v_2 \in Y_2} \left( J_2(u_1, v_2) + \frac{\alpha}{2} \|u_2 - v_2\|^2 \right) + \varepsilon \end{array} \right\}$$

and

$$N'_{\alpha,\varepsilon} = \left\{ \begin{array}{l} (u_1, u_2) \in Y \ : \ J_1(u_1, u_2) + J_2(u_1, u_2) \le \\ \inf\limits_{(v_1, v_2) \in Y} \left( J_1(v_1, u_2) + J_2(u_1, v_2) + \frac{\alpha}{2} \left( \|u_1 - v_1\|^2 + \|u_2 - v_2\|^2 \right) \right) + \varepsilon \end{array} \right\}.$$

In [30], an example, for $\alpha = 0$, proves that, generally, these two sets do not coincide. However, for what concerning well-posedness, both of them can be used. In fact, the following result holds:

**Proposition 3.1** *We have:*

$$\lim_{\varepsilon \to 0} \mathrm{diam} N_{\alpha,\varepsilon} = 0, \tag{3}$$

*if and only if*

$$\lim_{\varepsilon \to 0} \mathrm{diam} N'_{\alpha,\varepsilon} = 0. \tag{4}$$

*Proof* It is sufficient to observe that $N_{\alpha,\varepsilon} \subseteq N'_{\alpha,2\varepsilon}$ and $N'_{\alpha,\varepsilon} \subseteq N_{\alpha,\varepsilon}$, for every positive real number $\varepsilon$.
Indeed, let $(u_1, u_2) \in N'_{\alpha,\varepsilon}$ and consider, for every $v_1 \in Y_1$ and every $v_2 \in Y_2$, the points $(u_1, v_2)$ and $(v_1, u_2)$. Then, the two inequalities in $N_{\alpha,\varepsilon}$ are fulfilled and the point $\mathbf{u} = (u_1, u_2) \in N_{\alpha,\varepsilon}$. The other inclusion is obvious.

**Theorem 3.1** *If the NEP associated to the game* $\Gamma = (Y_1, Y_2, J_1, J_2)$ *is* $\alpha$*-well-posed, then*

$$N_{\alpha,\varepsilon} \neq \emptyset \ \forall \varepsilon > 0 \quad \textit{and} \quad \lim_{\varepsilon \to 0} \operatorname{diam} N_{\alpha,\varepsilon} = 0.$$

*Moreover, if the sets* $Y_1$ *and* $Y_2$ *are closed, convex and the following assumptions are satisfied*:

(i) *the function* $J_1 + J_2$ *is lower semicontinuous on* $Y$;
(ii) *for every* $v_2 \in Y_2$*, the function* $J_1(\cdot, v_2)$ *is convex and upper semicontinuous on* $Y_1$*; for every* $v_1 \in Y_1$*, the function* $J_2(v_1, \cdot)$ *is convex and upper semicontinuous on* $Y_2$*; then, the converse holds.*

*Proof* If the NEP is $\alpha$-well-posed, the set $\mathcal{N}$ of Nash equilibria is a singleton; since $N_{\alpha,\varepsilon} \supseteq \mathcal{N}$, it is nonempty for every $\varepsilon > 0$. If we assume that $\lim_{\varepsilon \to 0} \operatorname{diam} N_{\alpha,\varepsilon} > \beta > 0$, for every sequence of positive real numbers $(t_n)_n$ decreasing to 0, we could find two sequences $(\mathbf{u}_n)_n$ and $(\mathbf{w}_n)_n$, such that $\mathbf{u}_n \in N_{\alpha,t_n}, \mathbf{w}_n \in N_{\alpha,t_n}$ and $\|\mathbf{u}_n - \mathbf{w}_n\| > \beta$, for $n$ sufficiently large. But, this is in contradiction with the assumption, since the two sequences are $\alpha-$approximating and should converge towards the unique solution.

Now, assume that $\lim_{\varepsilon \to 0} \operatorname{diam} N_{\alpha,\varepsilon} = 0$ and let $(\mathbf{u}_n)_n$ be a sequence $\alpha$-approximating for the NEP. Then there exists a sequence of positive real numbers $(\eta_n)_n$ decreasing to 0, such that

$$J_1\left(u_1^n, u_2^n\right) \leq J_1\left(v_1, u_2^n\right) + \eta_n + \frac{\alpha}{2}\left\|u_1^n - v_1\right\|^2, \quad \forall v_1 \in Y_1, \quad \forall\, n \in N,$$

$$J_2\left(u_1^n, u_2^n\right) \leq J_2\left(u_1^n, v_2\right) + \eta_n + \frac{\alpha}{2}\left\|u_2^n - v_2\right\|^2, \quad \forall v_2 \in Y_2, \quad \forall\, n \in N.$$

By the assumption, the sequence $(\mathbf{u}_n)_n$ is a Cauchy sequence and has to converge to $\mathbf{u}_\text{o} \in Y$, which, in light of conditions $(i)$ and $(ii)$, satisfies the following:

$$J_1\left(u_1^\text{o}, u_2^\text{o}\right) + J_2\left(u_1^\text{o}, u_2^\text{o}\right) \leq \liminf_n \,(J_1 + J_2)\left(u_1^n, u_2^n\right) \leq \limsup_n \Big(J_1\left(v_1, u_2^n\right) + \eta_n$$
$$+\frac{\alpha}{2}\left\|u_1^n - v_1\right\|^2\Big) + \limsup_n \Big(J_2\left(u_1^n, v_2\right) + \eta_n + \frac{\alpha}{2}\left\|u_2^n - v_2\right\|^2\Big) \leq J_1\left(v_1, u_2^\text{o}\right)$$
$$+J_2\left(u_1^\text{o}, v_2\right) + \frac{\alpha}{2}(\left\|u_1^\text{o} - v_1\right\|^2 + \left\|u_2^\text{o} - v_2\right\|^2) \forall v_1 \in Y_1, \forall v_2 \in Y_2.$$

Taking $v_1 = u_1^\text{o}$ one gets:

$$J_2\left(u_1^\text{o}, u_2^\text{o}\right) \leq J_2\left(u_1^\text{o}, v_2\right) + \frac{\alpha}{2}\left\|u_2^\text{o} - v_2\right\|^2 \quad \forall v_2 \in Y_2,$$

while taking $v_2 = u_2^\text{o}$ one gets:

$$J_1\left(u_1^\text{o}, u_2^\text{o}\right) \leq J_1\left(v_1, u_2^\text{o}\right) + \frac{\alpha}{2}\left\|u_1^\text{o} - v_1\right\|^2 \quad \forall v_1 \in Y_1.$$

Now, it takes only to prove that $\left(u_1^\text{o}, u_2^\text{o}\right)$ is a Nash equilibrium for the game $(Y_1, Y_2, J_1, J_2)$. To this end, consider a point $\mathbf{w} = (w_1, w_2) \in Y$ and, for every $t \in [0, 1]$, the point $\mathbf{w}_t = t\mathbf{w} + (1 - t)\mathbf{u}_\text{o}$. Since $\mathbf{w}_t \in Y$, one has:

$$J_1\left(u_1^\text{o}, u_2^\text{o}\right) \leq J_1\left(w_1^t, u_2^\text{o}\right) + \frac{\alpha}{2}\left\|u_1^\text{o} - w_1^t\right\|^2 \quad \forall t \in ]0, 1].$$

$$J_2\left(u_1^\text{o}, u_2^\text{o}\right) \leq J_2\left(u_1^\text{o}, w_2^t\right) + \frac{\alpha}{2}\left\|u_2^\text{o} - w_2^t\right\|^2 \quad \forall t \in ]0, 1]$$

and, from the convexity assumptions:

$$J_1\left(u_1^{\mathrm{o}}, u_2^{\mathrm{o}}\right) \le t J_1\left(w_1, u_2^{\mathrm{o}}\right) + (1-t) J_1\left(u_1^{\mathrm{o}}, u_2^{\mathrm{o}}\right) + \frac{\alpha}{2} t^2 \left\| u_1^{\mathrm{o}} - w_1 \right\|^2 \quad \forall t \in ]0,1],$$

$$J_2\left(u_1^{\mathrm{o}}, u_2^{\mathrm{o}}\right) \le t J_2\left(u_1^{\mathrm{o}}, w_2\right) + (1-t) J_2\left(u_1^{\mathrm{o}}, u_2^{\mathrm{o}}\right) + \frac{\alpha}{2} t^2 \left\| u_2^{\mathrm{o}} - w_2 \right\|^2 \quad \forall t \in ]0,1].$$

Then, dividing by $t$:

$$J_1\left(u_1^{\mathrm{o}}, u_2^{\mathrm{o}}\right) \le J_1\left(w_1, u_2^{\mathrm{o}}\right) + \frac{\alpha}{2} t \left\| u_1^{\mathrm{o}} - w_1 \right\|^2 \quad \forall t \in ]0,1],$$

$$J_2\left(u_1^{\mathrm{o}}, u_2^{\mathrm{o}}\right) \le J_2\left(u_1^{\mathrm{o}}, w_2\right) + \frac{\alpha}{2} t \left\| u_2^{\mathrm{o}} - w_2 \right\|^2 \quad \forall t \in ]0,1],$$

which imply, for $t$ converging to zero, that $\mathbf{u}_{\mathrm{o}}$ is a Nash equilibrium for $(Y_1, Y_2, J_1, J_2)$.

**Remark 3.1** We have implicitly proved that, under the above assumptions, a point $\mathbf{u}_{\mathrm{o}}$ is a Nash equilibrium for $(Y_1, Y_2, J_1, J_2)$ if and only if it satisfies the following two conditions:

$$J_1\left(u_1^{\mathrm{o}}, u_2^{\mathrm{o}}\right) \le J_1\left(v_1, u_2^{\mathrm{o}}\right) + \frac{\alpha}{2} \left\| u_1^{\mathrm{o}} - v_1 \right\|^2 \quad \forall v_1 \in Y_1,$$

$$J_2\left(u_1^{\mathrm{o}}, u_2^{\mathrm{o}}\right) \le J_2\left(u_1^{\mathrm{o}}, v_2\right) + \frac{\alpha}{2} \left\| u_2^{\mathrm{o}} - v_2 \right\|^2 \quad \forall v_2 \in Y_2.$$

A characterization of $\alpha-$well-posedness in the generalized sense can be also given using the Kuratowski noncompactness measure $\mu$ instead of the diameter. More precisely, the following result holds.

**Theorem 3.2** *If the* NEP *is $\alpha$-well-posed in the generalized sense, then*

$$\forall \varepsilon > 0 \quad N_{\alpha,\varepsilon} \neq \emptyset \quad \text{and} \quad \lim_{\varepsilon \to 0} \mu(N_{\alpha,\varepsilon}) = 0.$$

*If the sets $Y_1$ and $Y_2$ are closed, convex and the following assumptions hold:*

(i) *the function $J_1 + J_2$ is lower semicontinuous on $Y$;*
(ii) *for every $v_2 \in Y_2$, the function $J_1(\cdot, v_2)$ is convex and upper semicontinuous on $Y_1$; for every $v_1 \in Y_1$, the function $J_2(v_1, \cdot)$ is convex and upper semicontinuous on $Y_2$, then the converse holds.*

*Proof* Assume that the NEP is $\alpha$-well-posed in the generalized sense and note that the set $\mathcal{N}$ of Nash equilibria is nonempty and compact. Indeed, if $(\mathbf{u}_n)_n$ is a sequence of Nash equilibria, it is also $\alpha$-approximating and, by the assumption, it contains a subsequence converging to a Nash equilibrium.

Now, we prove that $\lim_n \mu(N_{\alpha,\varepsilon_n}) = 0$ for every sequence $(\varepsilon_n)_n$ decreasing convergent to 0. First, we observe that $\mu(\mathcal{N}) = 0$, since the set $\mathcal{N}$ is compact, and that $N_{\alpha,\varepsilon_n}$ is bounded for all $n$, since the NEP is $\alpha$-well-posed in the generalized sense.

Then, being $\mathcal{N} \subseteq N_{\alpha,\varepsilon_n}$, from relation (2) at the end of Sect. 2 we infer:

$$\begin{aligned} \mu(N_{\alpha,\varepsilon_n}) &\le 2h(N_{\alpha,\varepsilon_n}, \mathcal{N}) + \mu(\mathcal{N}) \\ &= 2h(N_{\alpha,\varepsilon_n}, \mathcal{N}) = 2 \sup_{\mathbf{u} \in N_{\alpha,\varepsilon_n}} d(\mathbf{u}, \mathcal{N}). \end{aligned}$$

Thus, it takes only to prove that

$$\lim_n \sup_{\mathbf{u}\in N_{\alpha,\varepsilon_n}} d(\mathbf{u},\mathcal{N}) \leq 0.$$

If $\lim_n \sup_{\mathbf{u}\in N_{\alpha,\varepsilon_n}} d(\mathbf{u},\mathcal{N}) > c > 0$ there exists a positive integer $m$ and $\mathbf{u}_n \in N_{\alpha,\varepsilon_n} \forall n \geq m$ such that $d(\mathbf{u}_n,\mathcal{N}) > c$. So, the sequence $(\mathbf{u}_n)_n$ could not have subsequences converging to points of $\mathcal{N}$, in contradiction with the assumption of $\alpha$-well-posedness.

Now, assume that $\lim_{\varepsilon\to 0} \mu(N_{\alpha,\varepsilon}) = 0$. Since the sets $N_{\alpha,\varepsilon}$, for $\varepsilon > 0$, are closed and nonempty and $N_{\alpha,\varepsilon} \subseteq N_{\alpha,\varepsilon'}$ whenever $\varepsilon < \varepsilon'$, their intersection $\widetilde{\mathcal{N}}$ is nonempty, compact and satisfies: $\lim_{\varepsilon\to 0} h(N_{\alpha,\varepsilon}, \widetilde{\mathcal{N}}) = 0$ (see, e.g. [16, Theorem, p. 412]). Since

$$\widetilde{\mathcal{N}} = \left\{ \begin{array}{l} (u_1,u_2) \in Y : J_1(u_1,u_2) \leq \inf\limits_{v_1\in Y_1} \left(J_1(v_1,u_2) + \frac{\alpha}{2}\|u_1 - v_1\|^2\right) \\ \qquad\qquad\qquad\; J_2(u_1,u_2) \leq \inf\limits_{v_2\in Y_2} \left(J_2(u_1,v_2) + \frac{\alpha}{2}\|u_2 - v_2\|^2\right) \end{array} \right\},$$

arguing as at the end of Theorem 3.1, it can be proved that $\widetilde{\mathcal{N}}$ coincides with the set $\mathcal{N}$ of Nash equilibria for $(Y_1, Y_2, J_1, J_2)$ (see also Remark 3.1).

Finally, it takes only to prove that every $\alpha$-approximating sequence $(\mathbf{u}_n)_n$ has a converging subsequence. Since $\mathbf{u}_n \in N_{\alpha,\varepsilon_n}$, one has:

$$\lim_n d(\mathbf{u}_n,\mathcal{N}) \leq \lim_n h(N_{\alpha,\varepsilon_n},\mathcal{N}) = 0.$$

The set $\mathcal{N}$ being compact, there exists a sequence $(\mathbf{z}_n)_n$, $\mathbf{z}_n \in \mathcal{N}$, such that $d(\mathbf{u}_n,\mathcal{N}) = \|\mathbf{u}_n - \mathbf{z}_n\|$. The sequence $(\mathbf{z}_n)_n$ has a subsequence converging to a point of $\mathcal{N}$ towards which the corresponding subsequence of $(\mathbf{u}_n)_n$ has to converge.

The next theorems give the links between the concept of $\alpha$-well-posedness for Nash equilibria and the concept of $\alpha$-well-posedness for variational inequalities.

**Theorem 3.3 (**see Proposition 4.3 in [21]) *Assume that the sets $Y_1$ and $Y_2$ are closed, convex and*:

(i) *for every $u_2 \in Y_2$, the function $J_1(\cdot,u_2)$ is convex and lower semicontinuous on $Y_1$*;
(ii) *$J_1$ is Gâteaux differentiable with respect to $u_1$ on $Y$*;
(iii) *for every $u_1 \in Y_1$, the function $J_2(u_1,\cdot)$ is convex and lower semicontinuous on $Y_2$*;
(iv) *$J_2$ is Gâteaux differentiable with respect to $u_2$ on $Y$.*

*Then, if the NEP is $\alpha$-well-posed, the VI defined by the pair $(A,Y)$, where the operator $A$ is defined by*:

$$\langle A\mathbf{u},\mathbf{v}\rangle = \left\langle \frac{\partial J_1}{\partial u_1}(u_1,u_2), v_1 \right\rangle + \left\langle \frac{\partial J_2}{\partial u_2}(u_1,u_2), v_2 \right\rangle$$

*is $\alpha$-well-posed.*

**Theorem 3.4** *Assume that $E_1$ and $E_2$ are real Hilbert spaces, the sets $Y_1$ and $Y_2$ are closed, convex and bounded and*:

(i) *the function $J_1$ is Gâteaux differentiable with respect to $u_1$ and $\frac{\partial J_1}{\partial u_1}$ is uniformly continuous on $Y$*;
(ii) *the function $J_2$ is Gâteaux differentiable with respect to $u_2$ and $\frac{\partial J_2}{\partial u_2}$ is uniformly continuous on $Y$*;

(iii) *for every* $v_2 \in Y_2$ *and every* $v_1 \in Y_1$, $J_1(\cdot, v_2)$ *and* $J_2(v_1, \cdot)$ *are pseudoconvex on* $Y_1$ *and* $Y_2$ *respectively.*
*Then, if the the variational inequality defined by the pair* $(A, Y)$ *is* $\alpha$*-well-posed, the NEP is* $\alpha$*-well-posed.*

*Proof* Consider the following function $F$ defined on $Y \times Y$ by:

$$F(\mathbf{u}, \mathbf{w}) = J_1(w_1, u_2) + J_2(u_1, w_2).$$

If $(\mathbf{u}_n)_n = \left(u_1^n, u_2^n\right)_n$ is a sequence $\alpha$-approximating for the NEP, there exists a sequence $(t_n)_n$ of positive real numbers, decreasing to 0, such that:

$$J_1\left(u_1^n, u_2^n\right) \leq J_1\left(w_1, u_2^n\right) + t_n + \frac{\alpha}{2}\left\|u_1^n - w_1\right\|^2 \quad \forall w_1 \in Y_1 \quad \forall\, n \in N,$$
$$J_2\left(u_1^n, u_2^n\right) \leq J_2\left(u_1^n, w_2\right) + t_n + \frac{\alpha}{2}\left\|u_2^n - w_2\right\|^2 \quad \forall w_2 \in Y_2 \quad \forall\, n \in N.$$

Therefore,

$$F(\mathbf{u}_n, \mathbf{u}_n) \leq F(\mathbf{u}_n, \mathbf{w}) + 2t_n + \frac{\alpha}{2}\left(\left\|u_1^n - v_1\right\|^2 + \left\|u_2^n - v_2\right\|^2\right) \forall\, \mathbf{w} \in Y$$

and we can apply, for every $n \in N$, the Ekeland variational Principle [12] to the function:

$$\mathbf{w} \longrightarrow F(\mathbf{u}_n, \mathbf{w}) - \frac{\alpha}{2}\left(\left\|u_1^n - w_1\right\|^2 + \left\|u_2^n - w_2\right\|^2\right)$$

obtaining that there exists a sequence $(\mathbf{w}_n)_n = \left(w_1^n, w_2^n\right)$ in $Y$ such that:

$$\|\mathbf{u}_n - \mathbf{w}_n\| \leq \sqrt{2t_n}$$
$$\left\langle \tfrac{\partial F}{\partial \mathbf{w}}(\mathbf{u}_n, \mathbf{w}_n), \mathbf{w}_n - \mathbf{z}\right\rangle \leq \sqrt{2t_n}\,\|\mathbf{w}_n - \mathbf{z}\| + \alpha\left(\left\langle u_1^n - w_1^n, w_1^n - z_1\right\rangle + \left\langle u_2^n - w_2^n, w_2^n - z_2\right\rangle\right)$$
$$\leq \sqrt{2t_n}\,\|\mathbf{w}_n - \mathbf{z}\| + \alpha\left(\left\|u_1^n - w_1^n\right\|\left\|w_1^n - z_1\right\| + \left\|u_2^n - w_2^n\right\|\left\|w_2^n - z_2\right\|\right) \quad \forall \mathbf{z} \in Y.$$

Then

$$\left\langle \frac{\partial F}{\partial \mathbf{w}}(\mathbf{u}_n, \mathbf{w}_n), \mathbf{w}_n - \mathbf{z}\right\rangle \leq k\sqrt{2t_n}\,\mathrm{diam}(Y) \quad \forall \mathbf{z} \in Y, \tag{5}$$

where $k$ is a positive number. This inequality implies that the sequence $(\mathbf{w}_n)_n$ is $\alpha$-approximating for the VI. In fact, being:

$$\langle A\mathbf{w}_n, \mathbf{w}_n - \mathbf{z}\rangle = \left\langle \frac{\partial J_1}{\partial w_1}(w_1^n, w_2^n), w_1^n - z_1\right\rangle + \left\langle \frac{\partial J_2}{\partial w_2}(w_1^n, w_2^n), w_2^n - z_2\right\rangle$$

and

$$\left\langle \frac{\partial F}{\partial \mathbf{w}}(\mathbf{u}_n, \mathbf{w}_n), \mathbf{w}_n - \mathbf{z}\right\rangle = \left\langle \frac{\partial J_1}{\partial w_1}(w_1^n, u_2^n), w_1^n - z_1\right\rangle + \left\langle \frac{\partial J_2}{\partial w_2}(u_1^n, w_2^n), w_2^n - z_2\right\rangle,$$

one has:

$$\langle A\mathbf{w}_n, \mathbf{w}_n - \mathbf{z}\rangle = \left\langle \tfrac{\partial F}{\partial \mathbf{w}}(\mathbf{u}_n, \mathbf{w}_n), \mathbf{w}_n - \mathbf{z}\right\rangle + \left\langle A\mathbf{w}_n - \tfrac{\partial F}{\partial \mathbf{w}}(\mathbf{u}_n, \mathbf{w}_n), \mathbf{w}_n - \mathbf{z}\right\rangle$$
$$= \left\langle \tfrac{\partial F}{\partial \mathbf{w}}(\mathbf{u}_n, \mathbf{w}_n), \mathbf{w}_n - \mathbf{z}\right\rangle + \left\langle \frac{\partial J_1}{\partial w_1}(w_1^n, w_2^n) - \frac{\partial J_1}{\partial w_1}(w_1^n, u_2^n), w_1^n - z_1\right\rangle$$
$$+ \left\langle \frac{\partial J_2}{\partial w_2}(w_1^n, w_2^n) - \tfrac{\partial J_2}{\partial w_2}(u_1^n, w_2^n), w_2^n - z_2\right\rangle \quad \forall n \in N \quad \forall \mathbf{z} \in Y.$$

Since $\frac{\partial J_1}{\partial u_1}$ and $\frac{\partial J_2}{\partial u_2}$ are uniformly continuous, $Y$ is bounded and (5) holds, one gets:

$$\langle A\mathbf{w}_n, \mathbf{w}_n - \mathbf{z}\rangle \le \eta_n \quad \forall n \in N \ \forall \mathbf{z} \in Y,$$

where $(\eta_n)_n$ is a sequence of positive real numbers converging to zero. Therefore, the sequence $\left(w_1^n, w_2^n\right)_n$ is 0−approximating (and consequently $\alpha$-approximating) for (VI).

Since (VI) is $\alpha$-well-posed, the sequence $\left(w_1^n, w_2^n\right)_n$ has to converge to the unique solution $\mathbf{u}_o$ to (VI), and the same occurs for the sequence $(\mathbf{u}_n)_n$. Then, the result follows from Proposition 2.1.

**Remark 3.2** It is worth noting that in finite dimensional spaces the uniform continuity of partial derivatives is ensured by their continuity since the sets $Y_1$ and $Y_2$ are compact.

Now, we use the above result in order to identify classes of $\alpha$-well-posed NEPs, by passing through the classes of $\alpha$-well-posed variational inequalities.
First, we recall the results on variational inequalities.

**Proposition 3.2** **(**see Propsition 3.8 in [21]) *Let $\alpha \ge 0$, $E = R^k$ and $K$ be a nonempty, convex and compact subset of $E$. If the operator $A\colon E \longrightarrow E^*$ is monotone and hemicontinuous, then $(VI)$ is $\alpha$-well-posed if and only if $(VI)$ has a unique solution.*

**Proposition 3.3** *Let $E_1 = R^{h_1}$ and $E_2 = R^{h_2}$. If the sets $Y_1$ and $Y_2$ are convex and compact and the following assumptions hold*:

(i) *for every $u_2 \in Y_2$, the function $J_1(\cdot, u_2)$ is convex on $Y_1$;*
(ii) *for every $u_1 \in Y_1$, the function $J_2(u_1, \cdot)$ is convex on $Y_2$;*
(iii) *the function $J_1$ is Gâteaux differentiable with respect to $u_1$ and $\frac{\partial J_1}{\partial u_1}$ is continuous on $Y$;*
(iv) *the function $J_2$ is Gâteaux differentiable with respect to $u_2$ and $\frac{\partial J_2}{\partial u_2}$ is continuous on $Y$;*
(v) *for every $(u_1, u_2) \in Y$ and every $(v_1, v_2) \in Y$ :*

$$\left\langle \frac{\partial J_1}{\partial u_1}(u_1, u_2) - \frac{\partial J_1}{\partial u_1}(v_1, v_2), u_1 - v_1 \right\rangle + \left\langle \frac{\partial J_2}{\partial u_2}(u_1, u_2) - \frac{\partial J_2}{\partial u_2}(v_1, v_2), u_2 - v_2 \right\rangle \ge 0,$$

*that is, the operator $A$ defined in Theorem 3.3 is monotone;*
*then the NEP is $\alpha$-well-posed if and only if there exists a unique Nash equilibrium $\mathbf{u}_o$.*

*Proof* In light of Proposition 2.1, if $\mathbf{u}_o$ is the unique Nash equilibrium, it is also the unique solution to the associate VI. Then, one can apply Proposition 3.2 proving that (VI) is $\alpha$-well-posed. Using Theorem 3.4 one can conclude that the NEP is $\alpha$-well-posed.

In infinite dimensional spaces we have:

**Proposition 3.4** *Let $E_1$ and $E_2$ be two real Hilbert spaces and $Y_1$ and $Y_2$ be nonempty, closed, convex and bounded sets. Assume that the following assumptions hold*:

(i) *for every $u_2 \in Y_2$, the function $J_1(\cdot, u_2)$ is convex on $Y_1$;*
(ii) *for every $u_1 \in Y_1$, the function $J_2(u_1, \cdot)$ is convex on $Y_2$;*
(iii) *the function $J_1$ is Gâteaux differentiable with respect to $u_1$ and $\frac{\partial J_1}{\partial u_1}$ is uniformly continuous on $Y$;*

(iv) *the function* $J_2$ *is Gâteaux differentiable with respect to* $u_2$ *and* $\frac{\partial J_2}{\partial u_2}$ *is uniformly continuous on* $Y$*;*

(v) *for every* $\mathbf{u} = (u_1, u_2) \in Y$ *and every* $\mathbf{v} = (v_1, v_2) \in Y$*:*

$$\left\langle \frac{\partial J_1}{\partial u_1}(u_1, u_2) - \frac{\partial J_1}{\partial u_1}(v_1, v_2), u_1 - v_1 \right\rangle + \left\langle \frac{\partial J_2}{\partial u_2}(u_1, u_2) - \frac{\partial J_2}{\partial u_2}(v_1, v_2), u_2 - v_2 \right\rangle \geq \beta \|\mathbf{u} - \mathbf{v}\|^2,$$

*that is, the operator* $A$ *defined in Theorem 3.3 is strongly monotone with modulus* $\beta$*. Then the NEP is* $\alpha$*-well-posed for every* $\alpha \leq 2\beta$*.*

*Proof* From the assumptions and Proposition 3.6 in [21] one infers that (VI) is $\alpha$-well-posed for every $\alpha \leq 2\beta$. Therefore, the result follows from Theorem 3.4.

An interesting class of games satisfying the assumption (v) of Propositions 3.3 and 3.4, which include some oligopolistic markets, is given by *potential* games, as introduced by Monderer and Shapley [27]. We recall that $(Y_1, Y_2, J_1, J_2)$ is a potential game if there exists a function $P: Y_1 \times Y_2 \longrightarrow R$ such that for all $\mathbf{u} \in Y$ and all $\mathbf{v}, \mathbf{w} \in Y$:

$$J_1(u_1, v_2) - J_1(w_1, v_2) = P(u_1, v_2) - P(w_1, v_2),$$
$$J_2(v_1, u_2) - J_2(v_1, w_2) = P(v_1, u_2) - P(v_1, w_2).$$

This implies (see, e.g. [4]) that there exist two functions $g_1: Y_2 \longrightarrow R$ and $g_2: Y_1 \longrightarrow R$ such that:

$$J_1(u_1, u_2) = P(u_1, u_2) + g_1(u_2) \quad \text{and} \quad J_2(u_1, u_2) = P(u_1, u_2) + g_2(u_2).$$

If the function $P$ is convex (respectively, strongly convex) and Gâteaux differentiable the operator $A$ defined in Proposition 2.1 is monotone (respectively, strongly monotone).

**Remark 3.3** Sufficient conditions for the monotonicity or strong monotonicity of the operator $A$, which involve the second derivatives of $J_1$ and $J_2$, can be obtained adapting the results in [34], but, in our opinion, condition (v) is simpler to be verified.

**Remark 3.4** If one considers a zero-sum game taking $E_1 = E_2$, $Y_1 = Y_2 = Z$ and $J_1 = J$, $J_2 = -J$, definitions and results for the $\alpha$-well-posedness of the saddle point problem $(SPP)$, for the game $(Z, Z, J, -J)$, can be obtained using the former ones.

## 4 $\alpha$-Well-posedness for optimization problems with Nash equilibrium constraints

Let $(X, \tau)$ be a topological space and $\alpha$ be a nonnegative real number. With the notations used in the Introduction, we consider the problem:

$$\text{(OPNEC)} \qquad \min_{x \in X} \min_{(u_1, u_2) \in T(x)} f(x, u_1, u_2),$$

where, for every $x \in X$, $T(x)$ is the Nash equilibria set of the parametric game in normal form $\Gamma(x) = (Q_1(x), Q_2(x), J_1(x, \cdot), J_2(x, \cdot))$ and $Q_1$, $Q_2$ are set-valued mappings from $X$ to $Y_1$ and $Y_2$, respectively.

In order to introduce the concept of $\alpha$-well-posedness for these problems, we have to introduce a suitable concept of $\alpha$-approximating sequences which takes into account the hierarchical nature of the problem. Here, in line with [9, 28], we require that the $\alpha$-approximating sequences are minimizing sequences for the upper level problem and, at the same time, their second coordinates are $\alpha$-approximating for the lower level problem. More precisely, first we extend the concept of $\alpha$-well-posedness to a family of parametric Nash equilibrium problems **(NEP)** $= \{(\text{NEP})(x), x \in X\}$ and next we pass to problem (OPNEC).

**Definition 4.1** *Let $x \in X$ and $(x_n)_n$ be a sequence converging to $x$. A sequence $(\mathbf{u}_n)_n = (u_n^1, u_n^2)_n$ is said to be $\alpha$-approximating for $(NEP)(x)$ (with respect to $(x_n)_n$) if:*

(i) $\mathbf{u}_n \in Q(x_n) = Q_1(x_n) \times Q_2(x_n) \forall\, n \in N$;
(ii) *there exists a sequence $(t_n)_n$, $t_n > 0$, decreasing to $0$ such that:*

$$J_1\left(x_n, u_1^n, u_2^n\right) \leq t_n + J_1\left(x_n, v_1, u_2^n\right) + \tfrac{\alpha}{2}\left\|u_1^n - v_1\right\|^2 \quad \forall v_1 \in Q_1(x_n)\ \forall n \in N;$$
$$J_2\left(x_n, u_1^n, u_2^n\right) \leq t_n + J_2\left(x_n, u_1^n, v_2\right) + \tfrac{\alpha}{2}\left\|u_2^n - v_2\right\|^2 \quad \forall v_2 \in Q_2(x_n)\ \forall n \in N.$$

**Definition 4.2** *The family (**NEP**) is said to be parametrically $\alpha$-well-posed if:*

(i) *for every $x \in X$ the (NEP)$(x)$ has a unique solution $\mathbf{u}_x$;*
(ii) *for every sequence $(x_n)_n$ converging to $x$, every $\alpha$-approximating sequence $(\mathbf{u}_n)_n$ for $(NEP)(x)$ (with respect to $(x_n)_n$) strongly converges to $\mathbf{u}_x$.*

**Definition 4.3** *The family (**NEP**) is said parametrically $\alpha$*-well-posed *in the generalized sense if:*

(i) *for every $x \in X$ the (NEP)$(x)$ has at least a solution;*
(ii) *for every sequence $(x_n)_n$ converging to $x$, every $\alpha$-approximating sequence $(\mathbf{u}_n)_n$ for (NEP)(x) (with respect to $(x_n)_n$) has a subsequence which strongly converges to a solution to $(NEP)(x)$.*

Now, we recall the definition of $\alpha$-well-posedness (respectively, $\alpha$-well-posedness in the generalized sense), introduced in [9] for a family of parametric VIs defined by the pair $(A(x,\cdot), H(x))$, where: $E$ is a reflexive Banach space, $A(x,\cdot)\colon u \in E \longrightarrow A(x,u) \in E^*$ is an operator, $H\colon x \in X \longrightarrow H(x) \subseteq E$ is a set-valued mapping.

**Definition 4.4** *Let $x \in X$ and $(x_n)_n$ be a sequence converging to $x$. A sequence $(u_n)_n$ is said to be $\alpha$-approximating for $(VI)(x)$ (with respect to $(x_n)_n$) if:*

(i) $u_n \in H(x_n) \forall\, n \in N$;
(ii) *there exists a sequence $(\varepsilon_n)_n$, $\varepsilon_n > 0$, decreasing to $0$, such that*

$$\langle A(x_n, u_n), u_n - v\rangle - \frac{\alpha}{2}\|u_n - v\|^2 \leq \varepsilon_n \quad \forall\, v \in H(x_n)\ \forall\, n \in N.$$

**Definition 4.5** *The family (**VI**) is termed parametrically $\alpha$-well-posed if:*

(i) *for every $x \in X$, $(VI)(x)$ has a unique solution $u_x$;*
(ii) *for every sequence $(x_n)_n$ converging to $x$, every $\alpha$-approximating sequence $(u_n)_n$ for $(VI)(x)$ (with respect to $(x_n)_n$) strongly converges to $u_x$.*

**Definition 4.6** *The family of (**VI**) is termed parametrically $\alpha$-well-posed in the generalized sense if:*

(i) *for every* $x \in X$, $(VI)(x)$ *has at least a solution;*
(ii) *for every sequence* $(x_n)_n$ *converging to* $x$, *every* $\alpha$*-approximating sequence* $(u_n)_n$ *for* $(VI)(x)$ (*with respect to* $(x_n)_n$) *has a subsequence strongly convergent to a solution to* $(VI)(x)$.

In [9] conditions under which a family of parametric VIs is parametrically $\alpha$-well-posed or parametrically $\alpha$-well-posed in the generalized sense have been presented.

The link between parametrically $\alpha$-well-posedness for NEPs and VIs is furnished by the following result.

**Proposition 4.1** *Let* $\alpha \geq 0$, $E_1$ *and* $E_2$ *be two real Hilbert spaces,* $Y_1 \subseteq E_1$ *and* $Y_2 \subseteq E_2$ *be nonempty, closed and convex. Assume that the following conditions hold*:

(i) *the range of the set-valued mapping* $Q$ *defined by* $Q(x) = Q_1(x) \times Q_2(x)$ *is bounded*;
(ii) *for every* $x \in X$ *and for every* $u_2 \in Q_2(x)$, *the function* $J_1(x, \cdot, u_2)$ *is convex on* $Q_1(x)$;
(iii) *the function* $J_1$ *is Gâteaux differentiable with respect to* $u_1$ *and* $\frac{\partial J_1}{\partial u_1}$ *is uniformly continuous on* $X \times Q(X)$;
(iv) *for every* $x \in X$ *and for every* $u_1 \in Q_1(x)$, *the function* $J_2(x, u_1, \cdot)$ *is convex on* $Q_2(x)$;
(v) *the function* $J_2$ *is Gâteaux differentiable with respect to* $u_2$ *and* $\frac{\partial J_2}{\partial u_2}$ *is uniformly continuous on* $X \times Q(X)$.

Then, the family of NEP is parametrically $\alpha$-well-posed (respectively, $\alpha$-well-posed in the generalized sense) if and only if the family (**VI**) of associate VIs (see Proposition 2.1) is parametrically $\alpha$-well-posed (respectively, $\alpha$-well-posed in the generalized sense).

*Proof* The result can be proved using arguments similar to those in Theorems 3.3, 3.4.

Now, we introduce the $\alpha$-well-posedness concept for (OPNEC).

**Definition 4.7** *A sequence* $(x_n, \mathbf{u}_n)_n = \left(x_n, u_1^n, u_2^n\right)_n$ *is* $\alpha$*-approximating for* (*OPNEC*) *if:*

(i) $\limsup_n f(x_n, \mathbf{u}_n) \leq \inf_{x \in X} \inf_{\mathbf{u} \in T(x)} f(x, \mathbf{u})$;
(ii) $\mathbf{u}_n \in Q_1(x_n) \times Q_2(x_n)$ *for every* $n \in N$ *and there exists a sequence of positive real numbers* $(t_n)_n$ *decreasing to* $0$ *such that:*

$$J_1\left(x_n, u_1^n, u_2^n\right) \leq t_n + J_1\left(x_n, v_1, u_2^n\right) + \tfrac{\alpha}{2}\left\|u_1^n - v_1\right\|^2 \quad \forall v_1 \in Q_1(x_n)\ \forall n \in N,$$
$$J_2\left(x_n, u_1^n, u_2^n\right) \leq t_n + J_2\left(x_n, u_1^n, v_2\right) + \tfrac{\alpha}{2}\left\|u_2^n - v_2\right\|^2 \quad \forall v_2 \in Q_2(x_n)\ \forall n \in N.$$

**Definition 4.8** *The OPNEC is* $\alpha$*-well-posed if there exists a unique solution* $(x_o, \mathbf{u}_o)$ *and every* $\alpha$*-approximating sequence* $(\tau \times s)$*−converges to* $(x_o, \mathbf{u}_o)$.

**Definition 4.9** *The OPNEC is* $\alpha$*-well-posed in the generalized sense if there exists at least a solution and every* $\alpha$*-approximating sequence has a subsequence which* $(\tau \times s)$*−converges to a solution.*

We recall also the analogous definitions for an OPVIC:

$$\text{(OPVIC)} \qquad \min_{x \in X} \min_{u \in T(x)} h(x, u),$$

where $h: X \times E \to R \cup \{+\infty\}$ is bounded from below, $H$ is a set-valued function from $X$ to $E$, and, for every $x \in X$, $A(x, \cdot)$ is an operator from $E$ to $E^*$, while $T(x)$ is the set of solutions to the parametric (VI)(x) defined by the pair $(A(x, \cdot), H(x))$.

**Definition 4.10** [9] *A sequence $(x_n, u_n)_n$ is said to be $\alpha$-approximating for $(OPVIC)$ if:*

(i) $\limsup_n h(x_n, u_n) \leq \inf_{(x,u) \in X \times E, u \in T(x)} h(x, u)$;

(ii) *there exists a sequence $(\varepsilon_n)_n$, $\varepsilon_n > 0$, decreasing to 0, such that $u_n \in T_{a,\varepsilon_n}(x_n)\ \forall\, n \in N$, that is:*

$$u_n \in H(x_n) \quad and \quad \langle A(x_n, u_n), u_n - v\rangle - \frac{\alpha}{2}\|u_n - v\|^2 \leq \varepsilon_n \quad \forall\, v \in H(x_n).$$

**Definition 4.11** [9] *An OPVIC is termed $\alpha$-well-posed if:*

(i) *it has a unique solution $(x_0, u_0)$;*

(ii) *every $\alpha$-approximating sequence $(x_n, u_n)_n$ $(\tau \times s)-$converges to $(x_0, u_0)$.*

**Definition 4.12** [9] *An OPVIC is termed $\alpha$-well-posed in the generalized sense if:*

(i) *it has at least a solution;*

(ii) *every $\alpha$-approximating sequence $(x_n, u_n)_n$ has a subsequence $(\tau \times s)-$converging to a solution to $(OPVIC)$.*

We recall also that a function $f: X \times E_1 \times E_2 \longrightarrow R \cup \{\infty\}$ is called equicoercive with respect to $\tau \times \sigma$, where $\sigma$ is a topology on $E = E_1 \times E_2$, if it satisfies the following condition:

- every sequence $(x_n, \mathbf{u}_n)_n$, such that $f(x_n, \mathbf{u}_n) \leq k\ \forall\, n \in N$, has a subsequence converging in $\tau \times \sigma$.

The connection between the $\alpha$-well-posedness for the problem (OPNEC) and (OPVIC) is given by the following statement:

**Theorem 4.1** *Under the same assumptions of Proposition 4.1, the problem $(OPNEC)$ is $\alpha$-well-posed (respectively, $\alpha$-well-posed in the generalized sense) if and only if the corresponding problem $(OPVIC)$ is $\alpha$-well-posed (respectively, $\alpha$-well-posed in the generalized sense).*

*Proof* Let $(x_n, \mathbf{u}_n)$ be an $\alpha$-approximating sequence for the problem (OPVIC), that is there exists a sequence $(\varepsilon_n)_n$, $\varepsilon_n > 0$, decreasing to 0, such that

$$\mathbf{u}_n \in Q_1(x_n) \times Q_2(x_n) \quad \text{and} \quad \langle A(x_n, \mathbf{u}_n), \mathbf{u}_n - \mathbf{v}\rangle - \frac{\alpha}{2}\|\mathbf{u}_n - \mathbf{v}\|^2 \leq \varepsilon_n \quad \forall\, \mathbf{v} \in Q_1(x_n) \times Q_2(x_n) \quad \forall\, n \in N,$$

where

$$\langle A(x_n, \mathbf{u}_n), \mathbf{u}_n - \mathbf{v}\rangle = \left\langle \frac{\partial J_1}{\partial u_1}(x_n, u_1^n, u_2^n), u_1^n - v_1 \right\rangle + \left\langle \frac{\partial J_2}{\partial u_2}(x_n, u_1^n, u_2^n), u_2^n - v_2 \right\rangle.$$

Then, taking first $v_1 = u_1^n$ and, after, $v_2 = u_2^n$ one gets:

$$\left\langle \frac{\partial J_1}{\partial u_1}(x_n, u_1^n, u_2^n), u_1^n - v_1 \right\rangle - \frac{\alpha}{2}\left\|u_1^n - v_1\right\|^2 \leq \varepsilon_n \quad \forall\, v_1 \in Q_1(x_n),$$

$$\left\langle \frac{\partial J_2}{\partial u_2}(x_n, u_1^n, u_2^n), u_2^n - v_2 \right\rangle - \frac{\alpha}{2}\left\|u_2^n - v_2\right\|^2 \leq \varepsilon_n \quad \forall\, v_2 \in Q_2(x_n).$$

Now, it takes only to use the convexity of $J_1(x_n, \cdot, u_2^n)$ and $J_2(x_n, u_1^n, \cdot)$ in order to obtain that the sequence $(x_n, \mathbf{u}_n)$ is $\alpha$-approximating for (OPNEC). Then, if the problem (OPNEC) is $\alpha$-well-posed the corresponding problem (OPVIC) is $\alpha$-well-posed.

Conversely, assume that the problem (OPVIC) is $\alpha$-well-posed and consider, as in Theorem 3.4, the following function $F$ defined on $X \times Y \times Y$ by:

$$F(x, \mathbf{u}, \mathbf{w}) = J_1(x, w_1, u_2) + J_2(x, u_1, w_2).$$

If $(x_n, \mathbf{u}_n)_n$ is a sequence $\alpha$-approximating for (OPNEC), there exists a sequence $(t_n)_n$ of positive real numbers, decreasing to 0, such that:

$$J_1\left(x_n, u_1^n, u_2^n\right) \leq J_1\left(x_n, w_1, u_2^n\right) + t_n + \tfrac{\alpha}{2}\left\|u_1^n - w_1\right\|^2 \quad \forall w_1 \in Q_1(x_n),\ \ \forall n \in N,$$
$$J_2\left(x_n, u_1^n, u_2^n\right) \leq J_2\left(x_n, u_1^n, w_2\right) + t_n + \tfrac{\alpha}{2}\left\|u_2^n - w_2\right\|^2 \quad \forall w_2 \in Q_2(x_n),\ \ \forall n \in N.$$

Therefore,

$$F(x_n, \mathbf{u}_n, \mathbf{u}_n) \leq F(x_n, \mathbf{u}_n, \mathbf{w}) + 2t_n + \frac{\alpha}{2}\left(\left\|u_2^n - w_2\right\|^2 + \left\|u_1^n - w_1\right\|^2\right) \forall \mathbf{w} \in Q_1(x_n) \times Q_2(x_n)$$

and we can apply, for every $n \in N$, the Ekeland variational Principle to the function:

$$\mathbf{w} \in Q(x_n) \longrightarrow F(x_n, \mathbf{u}_n, \mathbf{w}) - \frac{\alpha}{2}\left(\left\|u_2^n - w_2\right\|^2 + \left\|u_1^n - w_1\right\|^2\right),$$

obtaining that there exists a sequence $(\mathbf{w}_n)_n = \left(w_1^n, w_2^n\right)$ in $Y$ such that:

$$\|\mathbf{u}_n - \mathbf{w}_n\| \leq \sqrt{2t_n},$$
$$\left\langle \tfrac{\partial F}{\partial \mathbf{w}}\left(x_n, \mathbf{u}_n, \mathbf{w}_n\right), \mathbf{w}_n - \mathbf{z}\right\rangle \leq \sqrt{2t_n}\,\|\mathbf{w}_n - \mathbf{z}\| + \alpha\left(\left\langle u_1^n - w_1^n, w_1^n - z_1\right\rangle + \left\langle u_2^n - w_2^n, w_2^n - z_2\right\rangle\right)$$
$$\leq \sqrt{2t_n}\,\|\mathbf{w}_n - \mathbf{z}\| + \alpha\left(\left\|u_1^n - w_1^n\right\|\left\|w_1^n - z_1\right\| + \left\|u_2^n - w_2^n\right\|\left\|w_2^n - z_2\right\|\right) \forall \mathbf{z} \in Q(x_n).$$

Then

$$\left\langle \frac{\partial F}{\partial \mathbf{w}}\left(x_n, \mathbf{u}_n, \mathbf{w}_n\right), \mathbf{w}_n - \mathbf{z}\right\rangle \leq k\sqrt{2t_n}\,\mathrm{diam}\,Q(x_n) \quad \forall \mathbf{z} \in Q(x_n), \tag{6}$$

where $k$ is a positive constant. Now, arguing as in Theorem 3.4, one proves that the sequence $\left(x_n, w_1^n, w_2^n\right)_n$ is 0−approximating for the problem (OPVIC) associated to (OPNEC). Indeed, one has:

$$\langle A(x_n, \mathbf{w}_n), \mathbf{w}_n - \mathbf{z}\rangle = \left\langle \frac{\partial F}{\partial \mathbf{w}}\left(x_n, \mathbf{u}_n, \mathbf{w}_n\right), \mathbf{w}_n - \mathbf{z}\right\rangle + \left\langle \frac{\partial J_1}{\partial w_1}(x_n, w_1^n, w_2^n) - \frac{\partial J_1}{\partial w_1}(x_n, w_1^n, u_2^n), w_1^n - z_1\right\rangle$$
$$+\left\langle \frac{\partial J_2}{\partial w_2}(x_n, w_1^n, w_2^n) - \frac{\partial J_2}{\partial w_2}(x_n, u_1^n, w_2^n), w_2^n - z_2\right\rangle \quad \forall n \in N\ \forall \mathbf{z} \in Q(x_n).$$

Since (6) and assumptions (i), (iii), (v) of Proposition 4.1 hold, there exists a sequence $(\eta_n)_n$ of positive real numbers converging to zero such that

$$\langle A(x_n, \mathbf{w}_n), \mathbf{w}_n - \mathbf{z}\rangle \leq \eta_n \quad \forall n \in N\ \forall \mathbf{z} \in Q(x_n).$$

Then, the sequence $\left(x_n, w_1^n, w_2^n\right)_n$ is 0−approximating (and consequently $\alpha$-approximating) for the problem (OPVIC) associated to (OPNEC) and has to converge to the unique solution $(x_o, \mathbf{u}_o)$ to (OPVIC), and the same occurs for the sequence $(x_n, \mathbf{u}_n)_n$. Since, in light of Proposition 2.1, $\mathbf{u}_o$ solves NEP$(x_o)$, $(x_o, \mathbf{u}_o)$ solves problem (OPNEC). So, the problem (OPNEC) is $\alpha$-well-posed. In a similar way one proves the result concerning $\alpha$-well-posedness in the generalized sense.

Now, we give a characterization of $\alpha$-well-posedness for (OPNEC) through the convergence to zero of the diameters of approximate solution sets. To this end we introduce the following set:

$$M_{\alpha,t,\delta} = \left\{ (x, \mathbf{u}) : \mathbf{u} \in N_{\alpha,t}(x) \text{ and } f(x, \mathbf{u}) \leq \inf_{x' \in X} \inf_{\mathbf{u}' \in T(x')} f(x', \mathbf{u}') + \delta \right\}.$$

**Theorem 4.2** *Assume that $(X, d)$ is a complete metric space, $E_1$ and $E_2$ are reflexive Banach spaces. If the problem* (OPNEC) *is $\alpha$-well-posed, then*

$$\forall t > 0 \; \forall \delta > 0 \;\; M_{\alpha,t,\delta} \neq \emptyset \;\; \text{and} \;\; \lim_{(t,\delta) \longrightarrow 0} \operatorname{diam} M_{\alpha,t,\delta} = 0.$$

*Moreover, assume that the following assumptions are satisfied:*

(i) *for every $x \in X$, the function $(J_1 + J_2)$ is lower semicontinuous on $\{x\} \times Q_1(x) \times Q_2(x)$;*
(ii) *for every $x \in X$ and for every $v_2 \in Q_2(x)$, the function $J_1(x, \cdot, v_2)$ is convex and upper semicontinuous on $Q_1(x)$;*
*for every $x \in X$, for every $v_1 \in Q_1(x)$, the function $J_2(x, v_1, \cdot)$ is convex and upper semicontinuous on $Q_2(x)$;*
(iii) *for every $x \in X$, the function $f$ is lower semicontinuous on $\{x\} \times Q_1(x) \times Q_2(x)$;*
(iv) *the set-valued mappings $Q_1$ and $Q_2$ are $s-$closed, $s-$lower semicontinuous and convex-valued.*

Then the converse holds.

*Proof* When the problem (OPNEC) is $\alpha$-well-posed, for every $t > 0$ and every $\delta > 0$ the set $M_{\alpha,t,\delta}$ is nonempty since it contains the unique solution $(x_o, \mathbf{u}_o)$. Moreover, if the diameters of such sets would not converge to 0, one could find two sequences $(t_n)_n$ and $(\delta_n)_n$ and a positive number $\eta$ such that:

$$\lim_{n \longrightarrow +\infty} t_n = 0 \quad \lim_{n \longrightarrow +\infty} \delta_n = 0 \quad \text{and} \quad \lim_{n \longrightarrow +\infty} \operatorname{diam} M_{\alpha,t_n,\delta_n} > \eta.$$

As a consequence, there would exist two sequences $(x_n, \mathbf{u}_n)_n$ and $(x'_n, \mathbf{u}'_n)_n$, which are $\alpha$-approximating for (OPNEC) and whose points have distance greater than a positive number $c$. This would contradict the assumption of $\alpha$-well-posedness.

On the other hand, assume that for every $t > 0$ and every $\delta > 0$ the set $M_{\alpha,t,\delta}$ is nonempty and $\lim_{(t,\delta)\longrightarrow 0} \operatorname{diam} M_{\alpha,t,\delta} = 0$. Let $(t_n)_n$ and $(\delta_n)_n$ be sequences of positive real numbers decreasing to 0 and let $(x_n, \mathbf{u}_n)_n$ be an $\alpha$-approximating sequence for $(OPNEC)$, that is $(x_n, \mathbf{u}_n) \in M_{\alpha,t_n,\delta_n}$. The sequence $(x_n, \mathbf{u}_n)_n$ is, by the assumption, a Cauchy sequence, so it has to converge to $(x_o, \mathbf{u}_o) \in X \times Y$. First, observe that the $s$-closedness of $Q_1$ and $Q_2$ implies that $\mathbf{u}_o \in Q_1(x_o) \times Q_2(x_o)$, while the lower semicontinuity of $f$ implies that $f(x_o, \mathbf{u}_o) \leq \inf_{x' \in X} \inf_{\mathbf{u}' \in T(x')} f(x', \mathbf{u}')$. In order to complete the proof it takes only to prove that $\mathbf{u}_o \in T(x_o)$, that is $(u_1^o, u_2^o)$ is a Nash equilibrium for the game $(Q_1(x_o), Q_2(x_o), J_1(x_o, \cdot), J_2(x_o, \cdot))$. To this end, let $(v_1, v_2) \in Q_1(x_o) \times Q_2(x_o)$. Since $Q_1$ and $Q_2$ are $s-$lower semicontinuous set-valued mappings, there exists a sequence $(v_1^n, v_2^n)_n$ strongly convergent to $(v_1, v_2)$ such that $(v_1^n, v_2^n) \in Q_1(x_n) \times Q_2(x_n)$. The fact that $\mathbf{u}_n \in N_{\alpha,t_n}(x_n)$ for every $n \in N$ amounts to:

$$J_1\left(x_n, u_1^n, u_2^n\right) \leq J_1\left(x_n, v_1^n, u_2^n\right) + t_n + \tfrac{\alpha}{2} \left\| u_1^n - v_1^n \right\|^2 \quad \forall n \in N,$$
$$J_2\left(x_n, u_1^n, u_2^n\right) \leq J_2\left(x_n, u_1^n, v_2^n\right) + t_n + \tfrac{\alpha}{2} \left\| u_2^n - v_2^n \right\|^2 \quad \forall n \in N.$$

From assumptions (i) and (ii) it follows that:

$$J_1\left(x_o, u_1^o, u_2^o\right) + J_2\left(x_o, u_1^o, u_2^o\right) \le J_1\left(x_o, v_1, u_2^o\right) + J_2\left(x_o, u_1^o, v_2\right) + \frac{\alpha}{2}\left(\left\|u_1^o - v_1\right\|^2 + \left\|u_2^o - v_2\right\|^2\right) \forall (v_1, v_2) \in Q_1(x_o) \times Q_2(x_o).$$

Then, considering the points $(u_1^o, v_2)$ and $(v_1, u_2^o)$, $v_1 \in Q_1(x_o)$, $v_2 \in Q_2(x_o)$, one obtains:

$$J_1\left(x_o, u_1^o, u_2^o\right) \le J_1\left(x_o, v_1, u_2^o\right) + \tfrac{\alpha}{2}\left\|u_1^o - v_1\right\|^2,$$
$$J_2\left(x_o, u_1^o, u_2^o\right) \le J_2\left(x_o, u_1^o, v_2\right) + \tfrac{\alpha}{2}\left\|u_2^o - v_2\right\|^2.$$

Finally, the result can be completed using the same arguments as in Theorem 3.1 (see also Remark 3.1).

In the remainder of the section we assume that $E_1$ and $E_2$ are real Hilbert spaces and we state some sufficient conditions for the $\alpha$-well-posedness or $\alpha$-well-posedness in the generalized sense of (OPNEC).

We point out that the set $T(x)$ of the solutions to (NEP)$(x)$ in general is not a singleton [34]; however, for the sake of completeness and taking into account that most of the applications concern the uniqueness case, we consider both the following situations:

- for every $x \in X$ (NEP)$(x)$ has a unique solution;
- there exists $x \in X$ such that (NEP)$(x)$ has not a unique solution.

**Case 1** For every $x \in X$ $(NEP)(x)$ has a unique solution.
In this situation the family **(NEP)** can be $\alpha$-well-posed, so we refer to Proposition 3.12 and Theorem 5.3 in [9] to prove the following:

**Theorem 4.3** *Let $Y_1$ and $Y_2$ be nonempty closed and convex sets, $f$ be sequentially lower semicontinuous and equicoercive on $(X \times E, \tau \times s)$. Assume that the problem $(OPNEC)$ has a unique solution (respectively, has at least a solution) and the following conditions hold*:

(i) *the set-valued mappings $Q_1$ and $Q_2$ are $w-$closed, $s-$lower semicontinuous, convex-valued and have bounded range*;
(ii) *for every $x \in X$ and for every $u_2 \in Q_2(x)$, the function $J_1(x, \cdot, u_2)$ is convex on $Q_1(x)$*;
(iii) *the function $J_1$ is Gâteaux differentiable with respect to $u_1$ and $\frac{\partial J_1}{\partial u_1}$ is uniformly continuous on $X \times Q(X)$*;
(iv) *for every $x \in X$ and for every $u_1 \in Q_1(x)$, the function $J_2(x, u_1, \cdot)$ is convex on $Q_2(x)$*;
(v) *the function $J_2$ is Gâteaux differentiable with respect to $u_2$ and $\frac{\partial J_2}{\partial u_2}$ is uniformly continuous on $X \times Q(X)$*;
(vi) *for every $x \in X$, for every $\mathbf{u} = (u_1, u_2) \in Y$ and $\mathbf{v} = (v_1, v_2) \in Y$ :*

$$\left\langle \frac{\partial J_1}{\partial u_1}(x, \mathbf{u}) - \frac{\partial J_1}{\partial u_1}(x, \mathbf{v}), u_1 - v_1 \right\rangle + \left\langle \frac{\partial J_2}{\partial u_2}(x, \mathbf{u}) - \frac{\partial J_2}{\partial u_2}(x, \mathbf{v}), u_2 - v_2 \right\rangle \ge \beta \|\mathbf{u} - \mathbf{v}\|^{\mathbf{2}},$$

*that is, the operator $A(x, \cdot)$, defined in Proposition 2.1, is strongly monotone with modulus $\beta$, uniformly with respect to $x$*;
(vii) *there exists a positive real number $k$ such that for every converging sequence $(x_n, \mathbf{u}_n)_n$ one has*

$$\left\| \frac{\partial J_1}{\partial u_1}(x_n, \mathbf{u}_n) \right\| + \left\| \frac{\partial J_2}{\partial u_2}(x_n, \mathbf{u}_n) \right\| \le k, \quad \text{for every} \quad n \in N.$$

*Then the problem* (OPNEC) *is $\alpha$-well-posed* (*respectively, $\alpha$-well-posed in the generalized sense*) *for every $\alpha \leq 2\beta$.*

*Proof* Following the assumptions, the operator $A(x, \cdot)$ and the set-valued mapping $Q$ satisfy the assumptions of Proposition 3.12 in [9]. Indeed, assumptions (i)–(iv) of Proposition 3.12 in [9] follow from assumptions (ii)–(vii), while (v) of Proposition 3.12 in [9] is a consequence of (i), since the boundeness of $Q(X)$ implies the $w-$subcontinuity of the set-valued mapping $Q$. In fact, the set $Q(X)$ is bounded so, in light of the Alaoglu theorem, it is relatively sequentially compact. The result follows applying Theorem 5.3 in [9] and Theorem 4.1.

**Remark 4.1** Let $E_1 = R^{h_1}$ and $E_2 = R^{h_2}$. Assume that the problem $(OPNEC)$ has a unique solution (respectively at least a solution), that all the assumptions of Theorem 4.3 except (vi) are satisfied and that the operator $A(x, \cdot)$ is monotone. Then the problem (OPNEC) is $\alpha$-well-posed (respectively, $\alpha$-well-posed in the generalized sense) for every $\alpha \geq 0$.

A class of games satisfying the assumption (vi) of Theorem 4.3 (respectively, of Remark 4.1) can be obtained, as observed in Sect. 3, considering parametric potential games where $P(x, \cdot, \cdot)$ is strongly convex on $Y$ uniformly with respect to $x$ (respectively, is convex on $Y$ for every $x \in X$).

**Case 2** There exists $x \in X$ such that the problem $(NEP)(x)$ does not have a unique solution. In this case, we have to consider parametric $\alpha$-well-posedness in the generalized sense of the family **(NEP)**, so, applying Propositions 3.12 and 3.16 in [9] we have:

**Theorem 4.4** *Let $Y_1$ and $Y_2$ be nonempty closed and convex sets, $f$ be sequentially lower semicontinuous and equicoercive on $(X \times E, \tau \times w)$. Assume that the problem $(OPNEC)$ has a unique solution (respectively has at least a solution), the conditions from (i) to (v) of Theorem 4.3 and the following hold*:

(vi) *for every $x \in X$, for every $(u_1, u_2) \in Y_1 \times Y_2$ and $(v_1, v_2) \in Y_1 \times Y_2$ :*

$$\left\langle \frac{\partial J_1}{\partial u_1}(x, u_1, u_2) - \frac{\partial J_1}{\partial u_1}(x, v_1, v_2), u_1 - v_1 \right\rangle + \left\langle \frac{\partial J_2}{\partial u_2}(x, u_1, u_2) - \frac{\partial J_2}{\partial u_2}(x, v_1, v_2), u_2 - v_2 \right\rangle \geq 0$$

*that is, the operator $A(x, \cdot)$, defined in Proposition 2.1, is monotone*;

(vii) *there exists a positive real number k such that for every converging sequence $(x_n, \mathbf{u}_n)_n$ one has*

$$\left\| \frac{\partial J_1}{\partial u_1}(x_n, \mathbf{u}_n) \right\| + \left\| \frac{\partial J_2}{\partial u_2}(x_n, \mathbf{u}_n) \right\| \leq k, \quad \text{for every} \quad n \in N.$$

*Then, for all $\alpha \geq 0$, every $\alpha$-approximating sequence for the problem $(OPNEC)$ weakly converges to the unique solution (respectively, has a subsequence weakly convergent to a solution), that is the problem $(OPNEC)$ is $\alpha$-well-posed (resppectively, $\alpha$-well-posed in the generalized sense) with respect to the weak convergence.*

Finally, in order to define a concept of well-posedness for the optimization problems with Saddle Points problem constraints (OPSPC), one can follow a scheme similar to the previous one. Namely, one can define:

- first, the concept of parametrically $\alpha$-approximating sequences for a family **(SPP)** of parametric $(SPP)(x)$, $x \in X$;
- second, the concept of $\alpha$-well-posedness for the problem (OPSPP).

Then, applying Theorem 4.4 or 4.5 one obtains the corresponding $\alpha$-well-posedness results for the (OPSPC).

## 5 Conclusions

Having in mind our previous results for VIs and OPVICs, we have investigated the $\alpha$-well-posedness for NEPs and for OPNECs. This choice has been motivated by the fact that most of the numerical methods for Nash equilibria are based on the associate variational inequality (see, e.g. [13]) and that the gap function introduced by Fukushima allows to construct numerical methods for VIs (see [14], Theorem 4.1). We point out that some results on Nash equilibria, as for example Theorems 3.1 and 3.2, have been obtained without using the associate variational inequality: such approach could be more investigated for our concept of $\alpha$-well-posedness as well as for other possible concepts of well-posedness.

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